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Bochner formulae for orthogonal G-structures on compact manifolds

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Abstract: A general technique is introduced for deriving Bochner type formulae on a compact riemannian manifold, relating its curvature tensor with the intrinsic torsion of a compatible (orthogonal) G-structure. The technique is illustrated for the groups $G = \mathrm{U}_n, \mathrm{SU}_n, \mathrm{G}_2$ and Spin_7, with various applications of the derived formulae in these cases.

Keywords: G-structure, intrinsic torsion, irreducible representation, curvature, Bochner formula.

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1. Introduction

Given an n-dimensional smooth manifold M and a subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$, a G-structure on M consists of a reduction of its structure group to G. For example, for $G = \mathrm{O}_n$ (the orthogonal group), this amounts to giving a riemannian metric on M. If we further reduce to a subgroup $G \subset \mathrm{O}_n$, we then say that we have an *orthogonal* G-structure (a structure which is compatible with the riemannian metric). For example, for $G = \mathrm{U}_n \subset \mathrm{O}_{2n}$ (the unitary group), this amounts to an *almost-hermitian structure* (an hermitian inner product at each tangent space).

An important invariant of a G-structure, playing a key role in Cartan's "method of equivalence" (classification of G-structures), is its *intrinsic torsion* tensor. It is a first order invariant measuring "flatness" (local-integrability) of the structure and its vanishing is equivalent to the existence of a torsionless G-connection on TM. See for example [3] for further information.

In this article we study a general scheme for obtaining a curvature obstruction to the existence of orthogonal G-structures on *compact* riemannian manifolds. This obstruction comes in the form of an integral formula relating the G-irreducible components of the intrinsic torsion tensor of the G-structure with G-invariants of the curvature tensor of the associated riemannian metric. We illustrate this technique for the groups $G = \mathrm{U}_n (n \geqslant 2), \mathrm{SU}_n (n \geqslant 3), G_2$ and Spin_7 (on manifolds of dimensions $2n$, $2n$, 7 and 8 resp.; SU_2 "belongs" to the Sp_n series).

For the first two groups (U_n and SU_n) our technique recaptures and extends several known results about almost-hermitian structures [7, 10, 18]. For the other two groups (G_2 and Spin_7) the results are apparently new.

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We have also applied the method to the groups $\mathrm{Sp}_k\mathrm{Sp}_1$ and Sp_k, but the calculations are more complicated and we leave these cases for a forthcoming article.

Here are some typical applications of the formulae derived in this paper:

– *Let* (M, g) *be a compact, conformally flat manifold with a compatible (orthogonal) almost-complex structure* J, *let* $\omega = g(J\cdot, \cdot)$ *be the corresponding "Kähler form" and denote by* s *the scalar curvature. Then*

(a) *If* J *is integrable, then* $\int_M s \geqslant 0$, *with equality if and only if* $d^*\omega = 0$.

(b) *If* ω *is symplectic* $(d\omega = 0)$ *then* $\int_M s \leqslant 0$, *with equality if and only if* J *is integrable (i.e. the structure is Kähler).*

Moreover, in dimension 4 *the result remains true for anti-self-dual manifolds (now the vanishing of the total scalar curvature is equivalent, in both cases, to the structure being Kähler).*

See Proposition 2 (Section 3.6 below). Part (a) was proved in [10].

– *Let* $M = \Gamma\backslash G/K$ *be a compact locally symmetric space with an orthogonal almost-complex structure* J. *Suppose that* M *is of non-compact type and that* J *is integrable, or that* M *is of compact type and the structure is symplectic. Then, in either case, the structure is in fact Kähler and descends from one of the* 2^k G*-invariant Kähler structures on* G/K, *where* G *is a product of* k *simple groups.*

See Proposition 4 (Section 3.6 below). Again, this extends results of [10] and [15].

– *Let* (M, g) *be a compact riemannian manifold of dimension* $\geqslant 6$ *with a compatible complex structure. Let* $i\kappa$ *be the curvature of the Levi-Civita connection on the canonical bundle of* M *and* s^* *the* $*$*-scalar curvature. Then*

$$\int_M \|\nabla\omega\|^2 = \int_M (2\langle\kappa, \omega\rangle - s^*).$$

In particular, such a manifold is Kähler if and only if $\int s^* = 2\int\langle\kappa, \omega\rangle$.

See Corollary 1 (Section 4.8 below). Although we have not found an explicit reference to this result in the literature, it might follow from results in [19].

– *Let* M *be a compact manifold of dimension* 7 *with a* G_2*-structure. Then its intrinsic torsion decomposes into four* G_2*-irreducible components,* $\tau = \tau_1 + \tau_7 + \tau_{14} + \tau_{27}$ *(indexed by the dimension of the irreducible subspace they belong to), satisfying*

$$\int_M 6\|\tau_1\|^2 + 5\|\tau_7\|^2 - \|\tau_{14}\|^2 - \|\tau_{27}\|^2 = \frac{2}{3}\int s,$$

where s *is the scalar curvature. Thus, for example, a compact riemannian manifold with a calibrated* G_2*-structure* ($d\phi = 0$, *where* ϕ *is the "fundamental"* 3*-form, or equivalently,* $\tau_1 = \tau_7 = \tau_{27} = 0$) *has non-positive total scalar curvature.*

See Corollary 3 (Section 5.5 below).

The rest of the article is organized as follows. The next section describes the technique we use, then each section treats in turn one of the four groups $G = \mathrm{U}_n$, SU_n, G_2 and Spin_7, with a few applications of the formulae obtained in each case.

2. The technique

Let $G \subset \mathrm{O}_n$ be the stabilizer subgroup of a p-form $\phi \in \Lambda^p(V^*)$ on $V = \mathbb{R}^n (p = 2, m, 3, 4$ for $G = U_m, \mathrm{SU}_m (n = 2m), G_2, \mathrm{Spin}_7$ resp.). Define a map $\Lambda^2 \otimes \Lambda^p \to \Lambda^p, \alpha \otimes \phi \mapsto \alpha \cdot \phi$, given by

$$(\theta \wedge \theta') \cdot \phi = \theta' \wedge [\mathrm{int}(\theta \otimes \phi)] - \theta \wedge [\mathrm{int}(\theta' \otimes \phi)], \qquad \theta, \theta' \in \Lambda^1, \quad \phi \in \Lambda^p,$$

where $\mathrm{int} : \Lambda^1 \otimes \Lambda^p \to \Lambda^{p-1}$ is "interior product" (contraction). Considering Λ^2 as the Lie algebra $\mathfrak{o}_n$ of O_n, this map is just the infinitesimal pull-back action of $\mathfrak{o}_n$ on Λ^p; thus, the Lie algebra $\mathfrak{g}$ of G is the kernel of $\cdot\phi : \alpha \mapsto \alpha \cdot \phi$. Note also that for $\alpha, \beta \in \Lambda^2, \alpha \cdot \beta = -\beta \cdot \alpha$, reflecting the fact that $\cdot$ corresponds to the Lie bracket under the identification of Λ^2 with $\mathfrak{o}_n$.

Now let M be an n-dimensional manifold with a G-structure and denote also by ϕ the associated p-form on M. The intrinsic torsion of such a G-structure can be identified with the covariant derivative $\nabla\phi \in \Lambda^1 \otimes \Lambda^p$. In fact,

Lemma 1. *$\nabla\phi$ lies in the sub-bundle $W = \Lambda^1 \otimes (\mathfrak{g}^\perp \cdot \phi) \subset \Lambda^1 \otimes \Lambda^p$. In particular, $W \cong \Lambda^1 \otimes \mathfrak{g}^\perp$.*

This is well-known. See for example [17].

Remark. We will use repeatedly the natural correspondence between G-representations and their associated bundles on a manifold with a G-structure. Thus, $V = \mathbb{R}^n$ corresponds to the tangent bundle, V^* to the cotangent bundle, $\Lambda^p(V^*)$ to $\Lambda^p M$ (so we can abbreviate safely both by Λ^p), an invariant subspace of a G-representation corresponds to a sub-bundle, a fixed element corresponds to a section of the associated bundle, etc.

Next, we decompose orthogonally the intrinsic torsion space $W = \Lambda^1 \otimes (\mathfrak{g}^\perp \cdot \phi)$, and consequently $\nabla\phi$, into G-irreducible components. These components carry interesting geometric information about the G-structure. For example, for $G = \mathrm{U}_n$ (almost-hermitian structure), $\nabla\phi$ has 4 irreducible components, the sum of certain two of them measuring the integrability of the associated almost-complex structure (the Nijenhuis tensor), the sum of certain three of them representing the local triviality of the corresponding almost-symplectic structure (the exterior derivative of the Kähler form), etc. See [13] for a complete description of the "16 types of U_n-structures".

Now we apply to ϕ the following integral formula, holding for any p-form on a *compact* Riemannian manifold (this follows, for example, from formulae WF I and WF II of [20, pp. 305–306]):

$$\int_M \|d\phi\|^2 + \|d^*\phi\|^2 - p!\, \|\nabla\phi\|^2 = \int_M \langle \widetilde{R}\phi, \phi \rangle. \tag{1}$$

Here $\widetilde{R}$ is the endomorphism on p-forms induced by the riemann curvature tensor $R \in \Lambda^2 \otimes \Lambda^2$ by applying (twice) the action of Λ^2 on Λ^p; thus, in components, with respect to a local frame of 1-forms $\{\theta_j\}$,

$$\widetilde{R}\phi = \sum_{i<j} R_{ij} \cdot [(\theta_i \wedge \theta_j) \cdot \phi].$$

Remark. We use the convention for the curvature tensor $R = \sum_{i<j} R_{ij} \otimes (\theta_i \wedge \theta_j)$ with $R_{ij} = [\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i, e_j]}$ (an anti-symmetric endomorphism interpreted as a 2-form), where $\{e_i\}$ is a frame dual to the θ_i.

Note that if $\nabla\phi$ vanishes (a "torsion-free" structure, or G-holonomy) one has $R \in \mathfrak{g} \otimes \mathfrak{g}$, hence $\widetilde{R}\phi = 0$.

We will also make use of a "twisted" version of this formula (in the U_n case) which looks exactly the same except that ϕ is a p-form on M with values in some vector bundle E with connection, where all objects $(\nabla, d, R, \ldots)$ are substituted by their twisted version [20, p. 430].

Next, $d\phi$ and $d^*\phi$ can be expressed in terms of $\nabla\phi$, $d\phi = \mathrm{alt}(\nabla\phi)$ and $d^*\phi = -\mathrm{int}(\nabla\phi)$, where $\mathrm{alt} : \Lambda^1 \otimes \Lambda^p \to \Lambda^{p+1}$ is alternation (extrior product). These are G-equivariant operations, so on the left hand side of formula (1) we obtain some quadratic form in the G-irreducible components of $\nabla\phi$. On the right hand side, the integrand is a G-invariant of the curvature tensor, so can be expressed in terms of the standard invariants, like the scalar curvature.

In executing the above plan for a given G, we make use of some simple techniques of representation theory, summarized in the following two lemmas.

Lemma 2. *Suppose that the intrinsic torsion space W, as a G-representation, is* multiplicity-free, *i.e., does not contain G-isomorphic irreducible subspaces* (*this occurs for all G except* SU_n, *in this article*) *and let $W = W_1 \oplus W_2 \oplus \cdots$ be the* (*unique*) *decomposition of W into mutually orthogonal, G-irreducible subspaces. Then there exist non-negative constants a_i, b_i such that for any $w_i \in W_i$, $\|\mathrm{alt}(w_i)\|^2 = a_i\|w_i\|^2$ and $\|\mathrm{int}(w_i)\|^2 = b_i\|w_i\|^2$. Hence, if we decompose $\nabla\phi$ into its irreducible components, $\nabla\phi = \sum_i (\nabla\phi)_i$, then*

$$\|d\phi\|^2 = \sum_i a_i \|(\nabla\phi)_i\|^2 \quad \text{and} \quad \|d^*\phi\|^2 = \sum_i b_i \|(\nabla\phi)_i\|^2.$$

This is essentially Schur's Lemma.

For the next lemma, recall that on a riemannian manifold, the curvature *operator* $\mathcal{R}$ is just the interpretation of the curvature tensor R as an endomorphism on 2-forms,

$$\mathcal{R}(\alpha) = -\sum_{i<j} \langle R_{ij}, \alpha\rangle \theta_i \wedge \theta_j, \qquad \alpha \in \Lambda^2,$$

(with the sign convention so as to make sure that $\mathcal{R}$ is a positive operator for the round sphere . . .).

Lemma 3. *Assume that the G-representation $\mathfrak{g}^\perp \subset \Lambda^2$ is multiplicity-free* (*this occurs for all four G studied in this article*) *and let $\mathfrak{g}^\perp = V_1 \oplus V_2 \oplus \cdots$ be the* (*unique*) *decomposition into G-irreducible mutually orthogonal subspaces. Let $c_j > 0$ be the homothety factors of $\cdot\phi : V_j \to W$ and denote by* $\mathrm{tr}(\mathcal{R}, V_j)$ *the trace of the "jj-block of $\mathcal{R}$", i.e. the trace of the endomorphism given by the restriction to V_j of the curvature operator $\mathcal{R}$, followed by orthogonal projection onto V_j. Then the integrand on the right-hand side of formula* (1) *is given by*

$$\langle \widetilde{R}\phi, \phi\rangle = \sum_j c_j \,\mathrm{tr}(\mathcal{R}, V_j).$$

Proof. The decomposition $\Lambda^2 = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$, $V_0 = \mathfrak{g}$, induces a "block" decomposition of the curvature tensor, $R = \sum_{i,j} r_{ij}$, $r_{ij} \in V_i \otimes V_j$. We claim that all but the r_{jj} terms, $j \geqslant 1$, are in the kernel of the map $R \mapsto \langle \widetilde{R}\phi, \phi\rangle$. Indeed, if $R = \alpha \otimes \beta$, with either α or $\beta \in \mathfrak{g}$, then $\langle \widetilde{R}\phi, \phi\rangle = \langle \alpha \cdot (\beta \cdot \phi), \phi\rangle = -\langle \beta \cdot \phi, \alpha \cdot \phi\rangle = 0$, since the map $\alpha\cdot : \Lambda^p \to \Lambda^p$ is skew-symmetric (being the derivative of an isometric O_n-action), and $\mathfrak{g}$ is the kernel of $\cdot\phi : \Lambda^2 \to \Lambda^p$. Similarly, if $R = \alpha \otimes \beta$ with $\alpha \in V_i$, $\beta \in V_j$, with i and j distinct and $\geqslant 1$, then again $\langle \widetilde{R}\phi, \phi\rangle = -\langle \beta \cdot \phi, \alpha \cdot \phi\rangle = 0$, since $V_i \cdot \phi$ and $V_j \cdot \phi$, being irreducible and non-isomorphic, are orthogonal subspaces of Λ^p. Finally, if $R = \alpha \otimes \beta$ with $\alpha, \beta \in V_j$, $j \geqslant 1$, then $\langle \widetilde{R}\phi, \phi\rangle = -\langle \beta \cdot \phi, \alpha \cdot \phi\rangle = -c_j\langle \beta, \alpha\rangle = c_j \operatorname{tr}(\mathcal{R}, V_j)$. $\square$

In the next four sections we illustrate the technique explained above for the groups $G = \mathrm{U}_n$, SU_n, G_2 and Spin_7: decomposition of W and $\mathfrak{g}^\perp$, calculation of the constants a_i, b_i and the curvature invariants $c_j \operatorname{tr}(\mathcal{R}, V_j)$, thus computing formula (1). The SU_n case is somewhat of an exception because Lemma 2 does not apply to it.

3. U_n

U_n is the stabilizer in O_{2n} of the "Kähler form" $\omega = \langle J\cdot, \cdot\rangle \in \Lambda^{1,1}$, where J is the standard almost-complex structure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (scalar multiplication by i). In terms of a unitary basis $z_1, \ldots, z_n$ for $\Lambda^{1,0}$, $\omega = i\sum_j z_j \wedge \bar{z_j}$. A manifold with a U_n-structure is often called an almost-hermitian manifold.

3.1. The decomposition of $\nabla\omega$

This was done in [13] and will be reviewed here briefly. In the decomposition $\Lambda^2 \otimes \mathbb{C} = \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}$, we have $\mathfrak{u}_n \otimes \mathbb{C} = \Lambda^{1,1}$ and $\mathfrak{u}_n^\perp \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2}$, so that $\mathfrak{u}_n^\perp$ is irreducible. Note that ω is the 2-form corresponding to J under the standard identification of anti-symmetric endomorphisms with 2-forms, hence for any (p, q)-form α we have

$$\omega \cdot \alpha = i(q - p)\,\alpha, \qquad \alpha \in \Lambda^{p,q}. \tag{2}$$

In particular, $\Lambda^{2,0}$ and $\Lambda^{0,2}$ are invariant under $\omega\cdot$, hence also $\mathfrak{u}_n^\perp$, the real part of $\Lambda^{2,0} \oplus \Lambda^{0,2}$. We thus get

$$\begin{aligned} W \otimes \mathbb{C} &= \Lambda^1 \otimes (\mathfrak{u}_n^\perp \cdot \omega) \otimes \mathbb{C} = (\Lambda^{1,0} \oplus \Lambda^{0,1}) \otimes (\Lambda^{2,0} \oplus \Lambda^{0,2}) \\ &= (\Lambda^{1,0} \otimes \Lambda^{2,0}) \oplus \Lambda^{1,2} + \text{conj.} \end{aligned}$$

Now,

$$\Lambda^{1,0} \otimes \Lambda^{2,0} + \text{conj.} = (W_1 \oplus W_2) \otimes \mathbb{C}, \qquad \Lambda^{1,2} + \text{conj.} = (W_3 \oplus W_4) \otimes \mathbb{C},$$

where:

W_1 is the real part of $\Lambda^{3,0}$;

W_2 is the real part of the image of $(\Lambda^{1,0} \otimes \Lambda^{1,0} \otimes \Lambda^{1,0})$ under the Young symmetrizer

$$(1 - (23))(1 + (12));$$

W_3 is the real part of the “primitive” part of $\Lambda^{1,2}$ (kernel of the contraction in the first and second entries);

$W_4 = \Lambda^1 \wedge \omega$.

Note that for $n = 1$: $W = 0$; for $n = 2$: $W_1 = W_3 = 0$ but W_2 and W_4 are non-zero and non-isomorphic, hence orthogonal; and that for $n \geqslant 3$ all four summands are non-zero and non-isomorphic, hence mutually orthogonal.

Denote by $\nabla\omega = (\nabla\omega)_1 + (\nabla\omega)_2 + (\nabla\omega)_3 + (\nabla\omega)_4$ the decomposition of $\nabla\omega$, where $(\nabla\omega)_i \in W_i$. In [13] one can get more information about the geometric meaning of the 4 components $(\nabla\omega)_i \in W_i$ and the vanishing of certain subgroups of the 4 components. We list here some of the better known possibilities:

Vanishing components of $\nabla\omega$	Name	Geometric meaning
all	Kähler	local holonomy $\subset \mathrm{U}_n$
1, 2	Hermitian	Integrability of the almost-complex structure
1,3,4	Almost-Kähler or symplectic	$d\omega = 0$
2,3,4	Nearly Kähler	$\nabla\omega = d\omega$
4	Cosymplectic or balanced	$d^*\omega = 0$

Table 1. Types of U_n-structures

3.2. *The left-hand side of formula* (1)

The W_i are mutually distinct (as U_n-representations), hence, according to Lemma 2, there are constants a_i such that

$$\|d\omega\|^2 = \|\mathrm{alt}(\nabla\omega)\|^2 = \sum_i \|\mathrm{alt}(\nabla\omega)_i\|^2 = \sum_i a_i \|(\nabla\omega)_i\|^2,$$

where a_i is the homothety factor of $\mathrm{alt} : W_i \to \Lambda^3$, i.e. $\|\mathrm{alt}(w_i)\|^2 = a_i \|w_i\|^2$ for every $w_i \in W_i$. Similarly,

$$\|d^*\omega\|^2 = \sum_i \|\mathrm{int}(\nabla\omega)_i\|^2 = \sum_i b_i \|(\nabla\omega)_i\|^2,$$

where the b_i are the homothety factors of $\mathrm{int} : W_i \to \Lambda^1$.

The next table summarizes the result of the computation of the homothety factors a_i, b_i and the elements $w_i \in W_i \otimes \mathbb{C}$ we used for computing them.

Summand	$w_i \in W_i \otimes \mathbb{C}$	$\|w_i\|^2$	$\|\mathrm{alt}(w_i)\|^2$	$\|\mathrm{int}(w_i)\|^2$	a_i	b_i
W_1	$z_1 \wedge z_2 \wedge z_3$	$\frac{1}{6}$	1	0	6	0
W_2	–	–	0	0	0	0
W_3	$z_1 \otimes (\bar{z}_2 \wedge \bar{z}_3)$	$\frac{1}{2}$	1	0	2	0
W_4	$\sum_{j=2}^n z_j \otimes (\bar{z}_j \wedge \bar{z}_1)$	$\frac{1}{2}(n-1)$	$n-1$	$(n-1)^2$	2	$2(n-1)$

Table 2. Calculation of the homothety factors a_i and b_i for U_n

Remarks. (i) To explain the zeros in the table, note that int : $W \to \Lambda^1$, but Λ^1 is irreducible and non-isomorphic to W_1, W_2 or W_3, hence $\mathrm{int}(W_1) = \mathrm{int}(W_2) = \mathrm{int}(W_3) = 0$. Similarly, Λ^3 does not contain a subspace isomorphic to W_2, hence $\mathrm{alt}(W_2) = 0$.

(ii) For the remaining entries, we pick elements w_i in $W_i \otimes \mathbb{C}$, $i = 1, 3, 4$, and compute $\mathrm{alt}(w_i)$ and $\mathrm{int}(w_i)$ and their norms. We use throughout a unitary basis $z_1, \ldots, z_n$ for $\Lambda^{1,0}$. Thus for example, we take $w_1 = z_1 \wedge z_2 \wedge z_3$. Then $\mathrm{alt}(w_1) = w_1$, hence $\|w_1\|^2 = \frac{1}{6}$ (as a tensor), $\|\mathrm{alt}(w_1)\|^2 = 1$ (as a 3-form) and so $a_1 = 6$.

It follows from this calculation that

$$\|d\omega\|^2 = 6\|(\nabla\omega)_1\|^2 + 2\|(\nabla\omega)_3\|^2 + 2\|(\nabla\omega)_4\|^2,$$

$$\|d^*\omega\|^2 = 2(n-1)\|(\nabla\omega)_4\|^2.$$

3.3. The right-hand side of formula (1) (*the curvature term*)

According to Lemma 3, since $\mathfrak{u}_n^\perp$ is irreducible, $\langle \widetilde{R}\omega, \omega\rangle = c\,\mathrm{tr}(R, \mathfrak{u}_n^\perp)$, where $c > 0$ is the homothety factor of $\cdot\omega : \mathfrak{u}_n^\perp \to \mathfrak{u}_n^\perp$. Since for $z_1 \wedge z_2 \in \mathfrak{u}_n^\perp \otimes \mathbb{C}$, $(z_1 \wedge z_2)\cdot\omega = 2i z_1 \wedge z_2$, one concludes that $c = 4$.

On the other hand, it is possible to express $\langle \widetilde{R}\omega, \omega\rangle$ in terms of the standard U_n-invariants, the scalar curvature $s = 2\,\mathrm{tr}(\mathcal{R})$ and the $*$-scalar curvature $s^* = 2\,\mathrm{tr}(J\mathcal{R})$.

Lemma 4. $\langle \widetilde{R}\omega, \omega\rangle = s - s^*$.

Proof. The splitting $\Lambda^2 = \mathfrak{u}_n \oplus \mathfrak{u}_n^\perp$ implies

$$\mathrm{tr}(\mathcal{R}) = \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n) + \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp), \qquad \mathrm{tr}(J\mathcal{R}) = \mathrm{tr}(J\mathcal{R}, \mathfrak{u}_n) + \mathrm{tr}(J\mathcal{R}, \mathfrak{u}_n^\perp).$$

Since $J = 1$ on $\Lambda^{1,1} = \mathfrak{u}_n \otimes \mathbb{C}$ and $J = -1$ on $\Lambda^{2,0} \oplus \Lambda^{0,2} = \mathfrak{u}_n^\perp \otimes \mathbb{C}$, we have that

$$\mathrm{tr}(J\mathcal{R}) = \mathrm{tr}(\mathcal{R}J) = \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n) - \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp).$$

Subtracting, we get $s - s^* = 2\,\mathrm{tr}(\mathcal{R}) - 2\,\mathrm{tr}(J\mathcal{R}) = 4\,\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp) = \langle \widetilde{R}\omega, \omega\rangle$. □

3.4. The U_n *formula*

Now denote by E_i the L_2-norm of the component $(\nabla\omega)_i$, so that $\int \|\nabla\omega\|^2 = E_1 + \cdots + E_4$. Plugging into formula (1) the information gathered above, the formula for the U_n case reduces to

$$2E_1 - E_2 + (n-1)E_4 = \frac{1}{2}\int s - s^*. \tag{3}$$

Note that E_3 does not appear and that $E_1 = E_3 = 0$ for $n = 2$ (i.e. real 4-manifolds).

3.5. Two homogeneous examples

3.5.1. The 6-sphere. The sphere S^6 with the round metric admits a homogeneous orthogonal almost-complex structure (G_2-invariant) which is “nearly-Kähler” ($\nabla\omega = d\omega$, see [11]). Since $s = 30$ and $s^* = 6$, formula (1) gives $E_1 = 6\,\mathrm{Vol}(S^6)$; by homogeneity, $\|d\omega\|^2 \equiv 6$ identically.

3.5.2. Product of odd spheres. $S^{2m+1} \times S^{2n+1} \subset \mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$ with the product of standard metrics admits a compatible $(U_{m+1} \times U_{n+1})$-homogeneous complex structure due to Hopf and Calabi-Eckmann. In fact, for $n = 0$ (so-called Hopf manifolds) $(\nabla\omega)_3 = 0$ as well (see [13]).

Working with the definition of J, we get $s - s^* = 4(m^2 + n^2)$, thus $(m+n)E_4 = 2(n^2 + m^2)\mathrm{Vol}(M)$; by homogenity, $\|(\nabla\omega)_4\|^2 \equiv 2(n^2 + m^2)/(m+n)$ identically .

3.6. Applications of the U_n *formula*

Some of the following applications appeared in [10] and [15], using similar techniques to ours (representation theory and integral formulae), although from a less general point of view. The literature on almost-hermitian manifolds is quite vast so it is possible that we have missed some other relevant references.

3.6.1. Complex structures and negative curvature.

Proposition 1. *On a compact riemannian manifold of dimension* $\geqslant 4$ *with a negative-definite curvature operator* ($\mathcal{R} < 0$) *any orthogonal almost-complex structure satisfies* $E_2 \neq 0$. *In particular, such a manifold cannot admit an orthogonal complex structure.*

Proof. Observe that the integrand on the right hand side of formula (3) is $\frac{1}{2}(s - s^*) = 2\,\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^{\perp}) < 0$. □

In particular, this result holds for a hyperbolic manifold, since its curvature operator is $-\mathrm{Id}_{\Lambda^2}$. This was also proved in [10] and [15].

Note that the compactness assumption is essential, because for the unit ball in $\mathbb{C}^n$ the standard hyperbolic and complex structures are compatible. In dimension 4 it is known that the *orthogonality* assumption is superfluous, i.e. a compact hyperbolic 4-manifold does not admit a complex structure (orthogonal or not), see [16]. For higher dimensions the problem is open. Another question, open as far as we know, is that of the existence of orthogonal almost-complex structures on compact hyperbolic manifolds. The only result in this direction that we are aware of is an unpublished proof of Kotschick that such examples do exist in dimension 4.

3.6.2. Conformally flat manifolds. A conformally flat manifold (i.e. locally conformal to euclidean space) is characterized by the vanishing of its Weyl tensor (for example [1, p. 60]).

Lemma 5. *On a conformally flat almost-hermitian* $2n$*-dimensional manifold* $s = (2n-1)\,s^*$.

Proof. The space of curvature-like tensors on $\mathbb{R}^{2n}$ with vanishing Weyl tensor is (O_{2n}-equivariantly) isomorphic to the space of quadratic forms on $\mathbb{R}^{2n}$ (essentially the space of Ricci tensors). The latter has a 1-dimensional space of U_n-invariants (by the Schur lemma, since $\mathbb{R}^{2n} = \mathbb{C}^n$ is an irreducible U_n-representation), hence s^* and s are proportional. It is thus sufficient to verify the identity for a single curvature operator of the said type, say for the identity operator on $\Lambda^2(\mathbb{R}^{2n})$ (the curvature operator of the sphere). We leave this simple verification to the reader. □

Consequently, on a conformally flat almost-hermitian compact manifold formula (3) reduces to

$$2E_1 - E_2 + (n-1)E_4 = \frac{n-1}{2n-1}\int_M s.$$

An immediate consequence of this is the following result.

Proposition 2. *Let M be a compact, conformally flat almost-hermitian manifold. Then*

(a) *If the structure is hermitian* (*i.e. J is integrable*), *then $\int_M s \geqslant 0$, with equality if and only if $d^*\omega = 0$;*

(b) *If the structure is symplectic* (*i.e. $d\omega = 0$*) *then $\int_M s \leqslant 0$, with equality if and only if the structure is Kähler.*

Proof. The assumption of part (a) is that $E_1 = E_2 = 0$, thus the inequality. Vanishing of the total scalar curvature further implies that $E_4 = 0$, and since $\|(\nabla\omega)_4\|^2$ is a multiple of $\|d^*\omega\|^2$ the result follows. Part (b) is proved similarly. □

Part (a) was proved in [10]. Observe that this again shows that on a compact real hyperbolic manifold there can be no orthogonal complex structure, since a hyperbolic manifold is conformally flat with $s < 0$.

In dimension 4 we get a stronger result, because the last lemma remains true for the larger class of *anti-self-dual manifolds*. Recall that an oriented riemannian 4-manifold is said to be anti-self-dual (ASD) if $W^+ \equiv 0$, where $W = W^+ \oplus W^-$ is the decomposition of the Weyl tensor relative to the eigenspaces of the Hodge * operator. Furthermore, in dimension 4, an hermitian manifold with $d^*\omega = 0$ is in fact Kähler (recall that $E_3 \equiv 0$ in dimension 4). Hence,

Proposition 3. *Let M be a compact, ASD almost-hermitian* 4*-manifold. Then*

(a) *If the structure is hermitian then $\int_M s \geqslant 0$, with equality if and only if the structure is Kähler.*

(b) *If the structure is symplectic then $\int_M s \leqslant 0$, with equality if and only if the structure is Kähler.*

Further information on the relation between hermitian geometry and the Weyl tensor may be found in [10] and the references therein.

3.6.3. Locally symmetric spaces. Some symmetric spaces G/K admit a G-invariant orthogonal complex structure (which is in fact Kähler), unique up to a sign if G is simple. These are the *hermitian symmetric spaces*. If G is a product of k simple groups then we get 2^k G-invariant orthogonal complex structures on G/K. We show next that on compact quotients of symmetric spaces of non-compact type (all irreducible factors are non-compact and non-euclidean) these are the only orthogonal complex structures. A dual statement for compact type spaces also holds.

Proposition 4. *Let M be an almost-hermitian manifold of dimension $\geqslant 4$ which is a compact connected quotient of a symmetric space $Z = G/K$ with k irreducible factors. Suppose that either*

(a) *the structure on M is hermitian and Z is of non-compact type*;

(b) *the structure on M is symplectic and Z is of compact type.*
Then Z is hermitian symmetric and the structure on M descends from one of the 2^k G-invariant Kähler structures on Z.

Note that once again, this result implies the impossibility of a hyperbolic hermitian structure on a compact manifold of dimension $\geqslant 4$, since (1) a hyperbolic manifold is a locally symmetric space of non-compact type, and (2) hyperbolic space is not hermitian symmetric in dimension > 2.

Part (a) was proved by [10] when G/K is irreducible and in [15] with the superfluous hypothesis that the irreducible factors of G/K should be different from the real hyperbolic plane.

For the proof given here we need, in addition to the Bochner formula (3), the following basic algebraic facts from the theory of symmetric spaces. We use the following standard notation: G/K is a symmetric space, $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K (resp.) and $\mathfrak{p} = \mathfrak{k}^{\perp}$ (with respect to the Killing form of $\mathfrak{g}$).

Lemma 6. *Suppose $J \in \mathrm{End}(\mathfrak{p})$ is an orthogonal complex structure such that its i-eigenspace $\mathfrak{p}^{1,0} \subset \mathfrak{p} \otimes \mathbb{C}$ is an* abelian *subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Then J is K-invariant, hence G/K is a hermitian symmetric space and J coincides with the value at $T_{[K]}G/K \cong \mathfrak{p}$ of one of the G-invariant orthogonal complex structures on G/K.*

This is essentially [6, lemma 4.8].

Lemma 7. *Let $\mathcal{R}$ be the curvature operator of a symmetric space G/K of compact or non-compact type. Then for any $X, Y, X', Y' \in \mathfrak{p} \cong T_{[K]}G/K$,*
$\langle \mathcal{R}(X \wedge Y), X' \wedge Y' \rangle = -\langle [X, Y], [X', Y'] \rangle$ *in the non-compact type case*;
$\langle \mathcal{R}(X \wedge Y), X' \wedge Y' \rangle = \langle [X, Y], [X', Y'] \rangle$ *in the compact type case.*

This follows from [14, theorem 4.2, p. 215].

Proof of Proposition 4. Fix a point $x \in M$ and identify $T_x M \cong \mathfrak{p}$. Next pick a unitary basis $\{z_i\}$ for $T_x^{1,0} M \cong \mathfrak{p}^{1,0}$ and use Lemma 7 above to calculate the integrand on the right hand side of formula (3) at x:

$$\tfrac{1}{2}(s - s^*) = 2\,\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^{\perp}) = 2 \sum_{i<j} \langle \mathcal{R}(z_i \wedge z_j), \overline{z_i \wedge z_j} \rangle = \pm 2 \sum_{i<j} \| [z_i, z_j] \|^2,$$

with the sign "–" in case (a) (non-compact type) and "+" in case (b) (compact type). In either case, we get different signs on the two sides of formula (3) and conclude that $\mathfrak{p}^{1,0}$ is an abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Lemma 6 above then implies that $Z = G/K$ is hermitian and the value of J at x comes from one of the 2^k G-invariant complex structures on G/K. By continuity, J corresponds to one of the G-invariant complex structures on Z. □

We single out two immediate consequences of the last proposition:

– Compact type: Proposition 4 determines all symplectic structures on $\mathbb{C}P^n$ (or products of them), compatible with the standard (Fubini-Study) metric.

– Non-compact type: products of compact hyperbolic manifolds of odd dimension do not admit an orthogonal complex structure (i.e. there is no negatively curved version of the Hopf-Calabi-Eckmann hermitian manifolds).

4. $\mathbf{SU}_n,\ n \geqslant 3$.

4.1. Definition of an SU_n*-structure*

The group $\mathrm{SU}_n,\ n \geqslant 2,$ is the stabilizer in U_n of the complex volume form $\psi = z_1 \wedge \ldots \wedge z_n \in \Lambda^{n,0}(V^*),\ \ V = \mathbb{C}^n = \mathbb{R}^{2n}$. Its (complexified) Lie algebra is the "primitive" part of $\Lambda^{1,1}$, i.e. the subspace orthogonal to ω. In fact,

Lemma 8. *For $n \geqslant 3$, SU_n is the stabilizer of ψ in O_{2n}.*

Proof. This must have appeared somewhere, but at any rate here is a sketch of a proof: first, check the claim on the Lie algebra level, i.e. $\alpha \cdot \psi = 0 \Longrightarrow \alpha \in \mathfrak{su}_n$ (divide into cases, according to the (p, q)-type of α). Next, $g^*\psi = \psi \Longrightarrow (g^*\omega) \cdot \psi = \omega \cdot \psi$ hence $g^*\omega - \omega \in \mathfrak{su}_n$. On the other hand, since $\mathfrak{su}_n = \mathrm{Ker}(\cdot\psi),\ g^*\psi = \psi \Longrightarrow g^*\mathfrak{su}_n = \mathfrak{su}_n \Longrightarrow g^*\mathfrak{su}_n^\perp = \mathfrak{su}_n^\perp$. Since $\omega \in \mathfrak{su}_n^\perp$, we get $g^*\omega - \omega \in \mathfrak{su}_n^\perp$, hence $g^*\omega = \omega$, so $g \in \mathrm{U}_n$. $\square$

Furthermore, it is easy to see that SU_n is the stabilizer of *any* non-zero element in $\Lambda^{n,0} \oplus \Lambda^{0,n}$; it turns out to be useful, for what follows, to choose this to be a real element, $\eta = \psi + \overline{\psi}$.

4.2. Decomposition of $\nabla\eta$

Under SU_n, the Lie algebra of U_n (real part of $\Lambda^{1,1}$) decomposes as $\mathfrak{u}_n = \mathfrak{su}_n \oplus \mathbb{R}\omega$, hence $\mathfrak{su}_n^\perp = \mathbb{R}\omega \oplus \mathfrak{u}_n^\perp$ (two SU_n-irreducible summands, distinct for $n \geqslant 3$). It follows that

$$W = \Lambda^1 \otimes (\mathfrak{su}_n^\perp \cdot \eta) = \Lambda^1 \otimes (\omega \cdot \eta) \ \oplus\ \Lambda^1 \otimes (\mathfrak{u}_n^\perp \cdot \eta).$$

Now $W_0 := \Lambda^1 \otimes (\omega \cdot \eta) \cong \Lambda^1$ is irreducible, where

$$\omega \cdot \eta = \omega \cdot (\psi + \overline{\psi}) = -in(\psi - \overline{\psi}) = n\eta^\perp, \qquad \eta^\perp := (\psi - \overline{\psi})/i.$$

As for the decomposition of $\Lambda^1 \otimes (\mathfrak{u}_n^\perp \cdot \eta)$ into irreducibles, we use the U_n-decomposition of $\Lambda^1 \otimes \mathfrak{u}_n^\perp$ into 4 irreducibles $W_1, \ldots, W_4$, and give their corresponding images the same names here (irreducible U_n representations remain irreducible upon restriction to SU_n). We thus obtain the decomposition

$$W = \Lambda^1 \otimes (\mathfrak{su}_n^\perp \cdot \eta) = W_0 \oplus \cdots \oplus W_4.$$

Remark. Note that we have two isomorphic summands: $W_0 \cong W_4 \cong \Lambda^1$. Thus, $\mathrm{alt}(\nabla\eta)_0$ and $\mathrm{alt}(\nabla\eta)_4$ need not be orthogonal (indeed they are not, in general), which complicates the expression for the left-hand side of formula (1) (appearance of "mixed terms").

4.3. The left-hand side of formula (1)

First, note that $*\eta = \pm\eta$, hence $\|d\eta\| = \|d^*\eta\|$. Thus,

$$\|d\eta\|^2 = \|d^*\eta\|^2 = \|\mathrm{int}(\nabla\eta)\|^2 = \sum_{j=0}^{4} \|\mathrm{int}(\nabla\eta)_j\|^2 + 2\langle \mathrm{int}(\nabla\eta)_0, \mathrm{int}(\nabla\eta)_4\rangle$$

$$= \sum_{i=0}^{4} a_i \|(\nabla\eta)_i\|^2 + 2\langle \mathrm{int}(\nabla\eta)_0, \mathrm{int}(\nabla\eta)_4\rangle,$$

with a_i the homothety factor of int (or alt) in W_j. The next table summarizes the calculation of the a_i. As a shorthand notation, we use $z_{1\bar{2}}$ for $z_1 \wedge \bar{z}_2$, etc.

Summand	$w_i \in W_i \otimes \mathbb{C}$	$\|w_i\|^2$	$\|\mathrm{int}(w_i)\|^2$	a_i
W_0	$z_1 \otimes \eta$	$2/n!$	1	$\frac{1}{2}n!$
W_1	$z_1 \otimes (z_{23}\cdot\eta) + z_2 \otimes (z_{31}\cdot\eta) + z_3 \otimes (z_{12}\cdot\eta)$	$6/n!$	12	$2\cdot n!$
W_2	$z_1 \otimes (z_{12}\cdot\eta)$	$2/n!$	1	$\frac{1}{2}n!$
W_3	$z_1 \otimes (z_{\bar{2}\bar{3}}\cdot\eta)$	–	0	0
W_4	$\sum_{k=2}^{n} z_k \otimes (z_{\bar{k}\bar{1}}\cdot\eta)$	$2(n-1)/n!$	$(n-1)^2$	$(n-1)\cdot\frac{1}{2}n!$

Table 3. Calculation of the homothety factors a_i for SU_n, $n \geqslant 3$

Clearly, an SU_n-structure induces an U_n-structure with its Kähler form ω, thus one expects to express $\nabla\omega$ in terms of $\nabla\eta$. This is done in the next lemma.

Lemma 9. *For any* SU_n*-structure with Kähler form* ω *and volume-forms* η *and* $\eta^\perp$ *as above, we have the orthogonal decomposition* $\nabla\eta = (\nabla\eta)_0 + \cdots + (\nabla\eta)_4$*, with*

(a) $(\nabla\eta)_0 = -\theta \otimes \eta^\perp$, *where* $\theta = -\frac{1}{2}\langle\nabla\eta, \eta^\perp\rangle = \langle\nabla\psi, \bar{\psi}\rangle/i$.

(b) $(\nabla\eta)_i = -\frac{1}{2}(\nabla\omega)_i \cdot \eta^\perp \in W_i$, $i = 1, \ldots, 4$, *where* $\cdot\eta^\perp : \Lambda^1 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^n$ *is given by* $\sigma \otimes \alpha \mapsto \sigma \otimes (\alpha \cdot \eta^\perp)$.

(c) $\|(\nabla\eta)_i\|^2 = \|(\nabla\omega)_i\|^2/n!$, $\quad i = 1, \ldots, 4$.

(d) $\langle \mathrm{int}(\nabla\eta)_0, \mathrm{int}(\nabla\eta)_4\rangle = \frac{1}{2}\langle\theta, d^*\omega\rangle$.

Remark. The 1-form $i\theta = \langle\nabla\psi, \bar{\psi}\rangle$ has the interpretation of the *connection form* of the canonical bundle $\Lambda^{n,0}$, relative to its section ψ.

Proof. (a) This is the usual projection formula on $\eta^\perp$ (note that $\|\eta\|^2 = 2$).

(b) Note first that by our definition of $\eta = \psi + \bar{\psi}$, $(\nabla\eta)_1 + \ldots + (\nabla\eta)_4 = (\nabla\psi)^{n-1,1} + \text{conj.}$, where $(\nabla\psi)^{n-1,1}$ is the component of $\nabla\psi$ in $\Lambda^1 \otimes \Lambda^{n-1,1}$. Now start from $\omega \cdot \psi = -in\psi$ (using equation 2) and apply ∇ to get $(\nabla\omega)\cdot\psi + \omega\cdot(\nabla\psi) = -in(\nabla\psi)$. By taking the (p, q)-decomposition of the last equation and applying again equation 2 to the (p, q)-components of $\nabla\psi$, we get $(\nabla\omega)\cdot\psi = -2i(\nabla\psi)^{n-1,1}$. Subtracting from the last equation its conjugate, we get $(\nabla\omega)\cdot\eta^\perp = -2[(\nabla\psi)^{n-1,1} + \text{conj.}]$, as claimed.

(c) The map $\cdot\eta^\perp : \mathfrak{u}_n^\perp \to \Lambda^n$ is a homothety onto its image (by Schur's lemma, since the domain is irreducible) and the homothety factor can be calculated by checking the effect on a single element (calculation omitted).

(d) We need to make use of the identity

$$\mathrm{int}(w\cdot\eta^{\perp}) = -\mathrm{int}[\mathrm{int}(w)\otimes\eta^{\perp}], \qquad w\in W_4\subset\Lambda^1\otimes\Lambda^2. \qquad (*)$$

Proof of (d): by the O_{2n}-equivariance of $\mathrm{int}:\Lambda^1\otimes\Lambda^n\to\Lambda^{n-1}$, $\alpha\cdot\mathrm{int}(\sigma\otimes\eta^{\perp}) = \mathrm{int}[(\alpha\cdot\sigma)\otimes\eta^{\perp}] + \mathrm{int}[\sigma\otimes(\alpha\cdot\eta^{\perp})]$, for all $\sigma\in\Lambda^1$, $\alpha\in\Lambda^2$. Now we claim that the map defined on $\Lambda^1\otimes\Lambda^2$ by the left-hand side of the last equation vanishes when restricted to W_4; by U_n-equivariance, it is enough to check this on a single element in $W_4\otimes\mathbb{C}$, say $\Sigma_{k=2}^n z_k\otimes z_{\bar k\bar 1}$. Indeed, $\Sigma_{k=2}^n z_{\bar k\bar 1}\cdot\mathrm{int}(z_k\otimes\eta^{\perp}) = 0$, since $\mathrm{int}(z_k\otimes\eta^{\perp})\in\Lambda^{0,n-1}$ and $\Lambda^{0,2}$ acts trivially on forms of type $(0,p)$. Now use the fact that $\alpha\cdot\sigma = \mathrm{int}(\sigma\otimes\alpha)$, for $\sigma\otimes\alpha\in\Lambda^1\otimes\Lambda^2$ and the identity $(*)$ follows.

Now using $\mathrm{int}[(\nabla\omega)_4] = -d^*\omega$ (see the U_n section) and item (b) (for $i=4$), we conclude that $\mathrm{int}[(\nabla\eta)_4] = -\frac{1}{2}\mathrm{int}(d^*\omega\otimes\eta^{\perp})$. Combined with item (a), we have

$$\langle\mathrm{int}(\nabla\eta)_0, \mathrm{int}(\nabla\eta)_4\rangle = \tfrac{1}{2}\langle\mathrm{int}(\theta\otimes\eta^{\perp}), \mathrm{int}(d^*\omega\otimes\eta^{\perp})\rangle.$$

Now the map $\Lambda^1\to\Lambda^{n-1}$, $\sigma\mapsto\mathrm{int}(\sigma\otimes\eta^{\perp})$ is SU_n−equivariant with an irreducible domain, hence a homothety onto its image, so it is enough to check its effect on a single element, say z_1, giving $\|\mathrm{int}(z_1\otimes\eta^{\perp})\| = \|\mathrm{int}(z_1\otimes\overline{\psi})\| = \|\bar z_2\wedge\ldots\wedge\bar z_n\| = 1$, i.e. we have an isometry, and so $\langle\mathrm{int}(\theta\otimes\eta^{\perp}), \mathrm{int}(d^*\omega\otimes\eta^{\perp})\rangle = \langle\theta, d^*\omega\rangle$, as required. □

Lemma 10.

$$\int\langle\mathrm{int}(\nabla\eta)_0, \mathrm{int}(\nabla\eta)_4\rangle = \frac{1}{2}\int\langle\kappa,\omega\rangle,$$

where $i\kappa = d\langle\nabla\psi,\overline{\psi}\rangle$ is the curvature 2-form of the connection induced by the Levi-Civita connection on $\Lambda^{n,0}$.

Proof. Integration by parts of item (d) of the previous lemma, □

4.4. The right hand side of formula (1) (*the curvature term*)

Lemma 11. $\langle\widetilde{R}\eta,\eta\rangle = \frac{1}{2}(s+s^*)$.

Proof. According to Lemma 3 of the introduction,

$$\langle\widetilde{R}\eta,\eta\rangle = c_1\,\mathrm{tr}(\mathcal{R},\mathbb{R}\omega) + c_2\,\mathrm{tr}(\mathcal{R},\mathfrak{u}_n^{\perp}),$$

with $c_1 = \|\omega\cdot\eta\|^2/\|\omega\|^2$ and $c_2 = \|\alpha\cdot\eta\|^2$ for a unitary $\alpha\in\Lambda^{2,0}$. We compute the terms appearing in this formula

c_1: we have $\omega\cdot\eta = -in(\psi-\overline{\psi})$, so $\|\omega\cdot\eta\|^2 = 2n^2$, $\|\omega\|^2 = n$, hence $c_1 = 2n$.

c_2: for $\alpha = z_1\wedge z_2$, $\|\alpha\|^2 = 1$ and $\alpha\cdot\eta = \alpha\cdot\overline{\psi} = (\alpha\cdot\bar\alpha)\wedge\bar z_3\wedge\ldots\wedge\bar z_n = (z_1\wedge\bar z_1 + z_2\wedge\bar z_2)\wedge\bar z_3\wedge\ldots\wedge\bar z_n$, so $\|\alpha\cdot\eta\|^2 = 2$ hence $c_2 = 2$.

$\mathrm{tr}(\mathcal{R},\mathfrak{u}_n^{\perp})$: in the U_n section we found $\mathrm{tr}(\mathcal{R},\mathfrak{u}_n^{\perp}) = \frac{1}{4}(s-s^*)$.

$\operatorname{tr}(\mathcal{R}, \mathbb{R}\omega)$: using the Bianchi identity,

$$\begin{aligned}\langle \mathcal{R}\omega, \omega\rangle &= \sum_{j,k}\langle \mathcal{R}z_{j\bar{k}}, z_{\bar{j}k}\rangle + \sum_{j,k}\langle \mathcal{R}z_{jk}, z_{\bar{k}\bar{j}}\rangle \\ &= \sum_{j,k}\langle \mathcal{R}Jz_{j\bar{k}}, \overline{z_{j\bar{k}}}\rangle + \sum_{j,k}\langle \mathcal{R}Jz_{jk}, \overline{z_{jk}}\rangle = \\ &= \operatorname{tr}(\mathcal{R}J, \Lambda^{1,1}) + 2\operatorname{tr}(\mathcal{R}J, \Lambda^{2,0}) = \operatorname{tr}(\mathcal{R}J) = \tfrac{1}{2}s^*,\end{aligned}$$

hence $\operatorname{tr}(\mathcal{R}, \mathbb{R}\omega) = \langle \mathcal{R}\omega, \omega\rangle/\|\omega\|^2 = s^*/2n$.

Combining all the above we get the desired result. □

4.5. *The* SU_n *formula*

Putting together all the information gathered so far in this section we arrive at

$$3E_1 - E_3 + (n-2)E_4 + 2\int_M \langle \kappa, \omega\rangle = \frac{1}{2}\int_M s + s^*. \tag{4}$$

where $E_j = \int_M \|(\nabla\omega)_j\|^2$ and $i\kappa$ is the curvature of the Levi-Civita connection on the canonical bundle.

4.6. *Extension to* U_n*-structures*

Observe that formula (4), as it now appears, makes sense for *any* U_n-structure, and that the only information eventually used about the SU_n-structure is that it *exists*, i.e. that we have a U_n-structure for which $c_1 = 0$ (the canonical bundle $\Lambda^{n,0}$ is topologically trivial). It is thus tempting to guess that the formula holds for any U_n-structure. Indeed,

Proposition 5. *Formula* (4) *holds for any* U_n*-structure*, $n \geqslant 3$.

This follows from the twisted version of formula (1) for the case of an n-form ϕ with values in the vector bundle $E =$ real part of $\Lambda^{n,0} \oplus \Lambda^{0,n}$, equipped with the connection induced by the Levi-Civita connection (by projection). We take $\phi = \eta \otimes \eta + \eta^\perp \otimes \eta^\perp$ (basically the inclusion map of E in Λ^n), then this section is defined globally, although η is defined only locally, and one checks easily that formula (1) applied to ϕ yields formula (4). Details of this calculation are routine and are left to the reader.

4.7. *The* U_n *examples revisited*

The S^6 *example.* Since it is of type W_1 the $(\nabla\eta)_4$ term vanishes hence also the crossed term. Since $s = 30$, $s^* = 6$, the equation is $3E_1 = (\frac{1}{2})36\,\mathrm{Vol}(S^6)$ so $E_1 = 6\,\mathrm{Vol}(S^6)$, consistent with the calculation with the U_n formula.

The $S^{2m+1} \times S^1$ example. We have already remarked that $S^{2m+1} \times S^1$ admits a U_n-structure of type W_4, from which we deduced that

$$m \int_M \|\nabla\omega\|^2 = \frac{1}{2} \int_M s - s^* = 2m^2 \mathrm{vol}(M).$$

For this example the formula derived in this section reads

$$(m-1) \int_M \|\nabla\omega\|^2 = \frac{1}{2} \int_M s + s^* - 2 \int_M \langle\kappa, \omega\rangle.$$

From which we deduce that,

$$\|\nabla\omega\|^2 = 2m = \langle\kappa, \omega\rangle.$$

4.8. Applications

For easy reference, let us collect both formulas (3) and (4), taking their sum and difference:

Proposition 6. *Let M be a compact almost-hermitian manifold of (real) dimension $2n$, $n \geqslant 3$. Then,*

$$5E_1 - E_2 - E_3 + (2n-3)E_4 = \int_M [s - 2\langle\kappa, \omega\rangle]$$

and

$$E_1 + E_2 - E_3 - E_4 = \int_M [s^* - 2\langle\kappa, \omega\rangle],$$

where $E_j = \int_M \|(\nabla\omega)_j\|^2$ and $i\kappa$ is the curvature of the canonical bundle.

One can use these formulas to characterize, via curvature conditions, different types of U_n-structures. For example,

Corollary 1. *Let M be a compact hermitian manifold of (real) dimension $\geqslant 6$. Then*

$$\int_M \|\nabla\omega\|^2 = \int_M 2\langle\kappa, \omega\rangle - s^*;$$

in particular, such a manifold is Kähler if and only if $\int s^* = 2\int\langle\kappa, \omega\rangle$.

Corollary 2. *Let M be an almost-hermitian compact manifold of (real) dimension $\geqslant 6$, with $d^*\omega = 0$ and $c_1 = 0$. Then*

$$5E_1 - E_2 - E_3 = \int_M s, \qquad \text{and} \qquad E_1 + E_2 - E_3 = \int_M s^*.$$

In particular,

- ([12]) *if M is nearly–Kähler and $c_1 = 0$, then $\int_M \|\nabla\omega\|^2 = \frac{1}{5}\int_M s = \int_M s^*$;*
- ([18]) *if M is almost-Kähler (i.e., symplectic) and $c_1 = 0$, then $\int_M \|\nabla\omega\|^2 = -\int_M s = \int_M s^*$;*
- ([18]) *if M is hermitian, $d^*\omega = 0$ and $c_1 = 0$, then $\int_M \|\nabla\omega\|^2 = -\int_M s = -\int_M s^*$.*

Proof. $c_1 = 0$ implies that κ is exact, hence, since $d^*\omega = 0$, integration by parts gives $\int_M \langle\kappa, \omega\rangle = 0$. Since $\|(\nabla\omega)_4\|$ is proportional to $\|d^*\omega\|$, the result follows from the above proposition. □

5. G_2

5.1. Definition of a G_2-structure

Let $V = \mathbb{R}^7$ with its standard euclidean structure, an orthonormal basis $\{e_1, \dots, e_7\}$ and dual basis $\{\theta_1, \dots, \theta_7\}$. Denote by $\Lambda^k := \Lambda^k(V^*)$ and basis elements by $\theta_{ij} := \theta_i \wedge \theta_j$, $\theta_{ijk} = \theta_i \wedge \theta_j \wedge \theta_k, \dots$ etc. The group $G_2 \subset O_7$ is the stabilizer of the 3-form

$$\phi = \theta_{124} + \theta_{235} + \theta_{346} + \theta_{457} + \theta_{561} + \theta_{672} + \theta_{713}.$$

(In fact, according to [2], G_2 is the stabilizer of ϕ in GL_7). A good way to remember this formula is to note that the 7 terms are obtained by cyclic permutations (mod 7) of the first term θ_{124}.

5.2. Decomposition of $\nabla\phi$

In the following decompositions the subscripts on irreducible G_2-representations denote their dimensions, omitting the detailed description of some irreducible spaces when their nature (other than their dimension) is immaterial for us. Fortunately, the G_2 irreducible representations that we meet here are distinguished by their dimensions alone. The following lemma is taken almost verbatim from [2] (see also [8] where the decomposition was first obtained).

Lemma 12. *Under G_2, we have the following decompositions into irreducible subspaces*:

$\Lambda^2 = \Lambda^2_{(14)} \oplus \Lambda^2_{(7)}$, *corresponding to* $\mathfrak{so}_7 = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$, *where* $\Lambda^2_{(14)} = \mathfrak{g}_2$ *is the kernel of* $\cdot\phi : \Lambda^2 \to \Lambda^3$, $\alpha \mapsto \alpha \cdot \phi$, *and* $\Lambda^2_{(7)} = \mathfrak{g}_2^\perp \cong \Lambda^1$ *is the image of* $\theta \mapsto \mathrm{int}(\theta \otimes \phi)$, $\theta \in \Lambda^1$.

$\Lambda^3 = \Lambda^3_{(1)} \oplus \Lambda^3_{(7)} \oplus \Lambda^3_{(27)}$, *where* $\Lambda^3_{(1)} = \mathbb{R}\phi$ *and* $\Lambda^3_{(7)} \cong \Lambda^1$ *is the image of* $\theta \mapsto \mathrm{int}[\theta \otimes (*\phi)]$, $\theta \in \Lambda^1$.

Next, we decompose

$$\Lambda^1 \otimes \Lambda^3_{(7)} \cong \Lambda^1 \otimes \Lambda^1 = \Lambda^2 \oplus \mathrm{S}^2 = \Lambda^2_{(14)} \oplus \Lambda^2_{(7)} \oplus \mathrm{S}^2_{(1)} \oplus \mathrm{S}^2_{(27)},$$

where we use the decomposition of S^2 (quadratic forms on V) into {multiples of the inner-product} $\oplus$ {traceless}.

It is easy to see that $\mathrm{S}^2_{(1)}$ consists of multiples of the invariant 4-form $*\phi$, using the following argument: since Λ^3 contains a single isomorphic copy of Λ^1, it follows (using Schur's lemma) that the subspace of fixed vectors in $\Lambda^1 \otimes \Lambda^3$ is 1-dimensional. On the other hand, $\Lambda^4 \subset \Lambda^1 \otimes \Lambda^3$, hence the fixed subspace in $\Lambda^1 \otimes \Lambda^3$, and hence in $\Lambda^1 \otimes \Lambda^3_{(7)}$, must consist precisely of the multiples of $*\phi$.

If we name the irreducible subspaces of W by their dimension, then we have

$$W = \Lambda^1 \otimes (\mathfrak{g}_2^\perp \cdot \phi) = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}.$$

5.3. The left-hand side of formula (1)

Denote by $\nabla\phi = (\nabla\phi)_1 + (\nabla\phi)_7 + (\nabla\phi)_{14} + (\nabla\phi)_{27}$ the decomposition of $\nabla\phi$, where $(\nabla\phi)_i \in W_i$. The W_i are mutually distinct G_2-representations, hence, according to Lemma 2 of the Introduction, there are constants a_i such that

$$\|d\phi\|^2 = \|\mathrm{alt}(\nabla\phi)\|^2 = \sum_i \|\mathrm{alt}(\nabla\phi)_i\|^2 = \sum_i a_i \|(\nabla\phi)_i\|^2,$$

where a_i is the homothety factor of alt : $W_i \to \Lambda^4$, i.e. $\|\mathrm{alt}(w_i)\|^2 = a_i\|w_i\|^2$ for every $w_i \in W_i$. Similarly,

$$\|d^*\phi\|^2 = \sum_i \|\mathrm{int}(\nabla\phi)_i\|^2 = \sum_i b_i \|(\nabla\phi)_i\|^2,$$

where the b_i are the homothety factors of int : $W_i \to \Lambda^2$.

The next table summarizes the result of the computation of the homothety factors a_i and b_i. We use the map $T : \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^3_{(7)},\ \theta \otimes \theta' \mapsto \theta \otimes \mathrm{int}[\theta' \otimes (*\phi)]$.

Summand	element $w_i \in W_i$	$\|w_i\|^2$	$\|\mathrm{alt}(w_i)\|^2$	$\|\mathrm{int}(w_i)\|^2$	a_i	b_i
W_1	$*\phi$	7/24	7	0	24	0
W_7	$T[\mathrm{int}(\theta_1 \otimes \phi)]$	4	36	48	9	12
W_{14}	$T(\theta_{12} + \theta_{36})$	8/3	0	8	0	3
W_{27}	$T(\theta_1 \otimes \theta_1 - \theta_2 \otimes \theta_2)$	4/3	4	0	3	0

Table 4: Calculation of the homothety factors a_i, b_i for G_2

We thus have

$$\|d\phi\|^2 = 24\|(\nabla\phi)_1\|^2 + 9\|(\nabla\phi)_7\|^2 + 3\|(\nabla\phi)_{27}\|^2,$$

$$\|d^*\phi\|^2 = 12\|(\nabla\phi)_7\|^2 + 3\|(\nabla\phi)_{14}\|^2.$$

Remark. It follows from these formulas that ϕ is *parallel* ($\nabla\phi = 0$) if it is harmonic ($d\phi = d^*\phi = 0$).

5.4. The right-hand side of formula (1) (*the curvature term*)

The curvature term turns out to be particularly simple in the G_2 case.

Lemma 13. *Let M be a* 7*-dimensional manifold with a* G_2*-structure, a* 3*-form* ϕ, *a Riemann curvature tensor R and scalar curvature s. Then* $\langle \widetilde{R}\phi, \phi\rangle = 2s$.

Proof. Let us prove first that the space of G_2-fixed curvature-type tensors is 1-dimensional: since Λ^2 decomposes into two non-isomorphic irreducible subspaces, the Schur lemma implies that the space of G_2-fixed elements in $S^2(\Lambda^2)$ is 2-dimensional. Now the G_2-fixed subspace of Λ^4 is 1-dimensional (multiples of $*\phi$), hence the G_2-fixed subspace of its orthogonal complement in $S^2(\Lambda^2)$ (this is exactly the space of curvature-type tensors) is also 1-dimensional, as required. It is therefore enough to verify the identity on a particular curvature-type tensor (for which the outcome is non-zero), say $R = \theta_{12} \otimes \theta_{12}$. This (simple) verification is left to the reader. □

5.5. The G_2 formula

Denote by $E_1, \ldots, E_{27}$ the L_2 norms of $(\nabla\phi)_1, \ldots, (\nabla\phi)_{27}$ (resp.), then substituting all of the above into formula (1), we get

$$6E_1 + 5E_7 - E_{14} - E_{27} = \frac{2}{3} \int s. \tag{5}$$

Corollary 3. *Let M be a compact, calibrated (i.e. $d\phi = 0$) G_2-manifold, then $\int s \leqslant 0$ with equality if and only if the local holonomy of M is contained in G_2.*

5.6. Examples

Hypersurfaces in $\mathbb{R}^8$. Consider $\mathbb{R}^8$ with its standard Spin_7-structure $\Phi \in \Lambda^4((\mathbb{R}^8)^*)$ (see next section). Then on any oriented hypersurface $M^7 \subset \mathbb{R}^8$ there is a G_2-structure defined by $\phi = \mathrm{int}_N \Phi$, where N is the unit normal on M given by the orientation. For example if M is a linear subspace (say $x_0 = 0$) then we get the standard ϕ on $\mathbb{R}^7$. If M is the unit sphere $S^7 \subset \mathbb{R}^8$ we get the homogeneous space Spin_7/G_2 with its (essentially unique) Spin_7-invariant G_2-structure.

For any hypersurface in $\mathbb{R}^8$ the above defined G_2-structure is of type $W_1 \oplus W_{27}$, see [8], and therefore the formula for them reduces to

$$6E_1 - E_{27} = \frac{2}{3} \int_M s.$$

An Aloff-Wallach space. Such a space is a homogeneous space of the form SU_3/U_1 equipped with a left invariant metric. In [5] it is shown that, for a certain choice of subgroup U_1, the 7-manifold SU_3/U_1 admits (a non-parallel) SU_3-invariant G_2-structure of type W_{27}. For it, our formula immediately implies that $\|\nabla\phi\|^2 = -\frac{2}{3}s$; in particular, this G_2-structure is associated to a left-invariant metric on SU_3 with $s < 0$.

Integrable G_2-structures. In [9] G_2-structures of type $W_1 \oplus W_7 \oplus W_{27}$ are studied; it is discussed there why these structures should be considered the G_2 analogues of integrable almost hermitian structures. A few examples are given.

Nearly-parallel G_2-structures. In a recent work of N. Hitchin (see math.DG/0107101 in http://xxx.lanl.gov), G_2-structures of type W_1 (called also nearly-parallel G_2 manifolds, or manifolds with weak-holonomy G_2) appear in relation with various natural variational problems. For such manifolds our formula (5) above gives $\int s = 9 \int \|\nabla\phi\|^2 \geqslant 0$.

Further examples and properties may be found in [8].

6. Spin_7

6.1. Definition of a Spin_7*-structure.*

Let $V = \mathbb{R}^8$ with the standard euclidean structure, an orthonormal basis $e_0, e_1, \dots, e_7$, and dual basis $\theta_0, \dots, \theta_7$. Denote by $\Lambda^k := \Lambda^k(V^*)$ and basis elements by $\theta_{ij} := \theta_i \wedge \theta_j$, $\theta_{ijk} = \theta_i \wedge \theta_j \wedge \theta_k, \dots$ etc.

The group $\mathrm{Spin}_7 \subset \mathrm{SO}_8$ is the stabilizer of the 4-form $\Phi = \theta_0 \wedge \phi + *\phi$. Here we are thinking of $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$, with ϕ and $*\phi$ on $\mathbb{R}^7$ (pulled-back to $\mathbb{R}^8$) as defined before for G_2.

6.2. Decomposition of $\nabla\Phi$

Like in the G_2 case, we denote representations by their dimensions and omit further information whenever is not used here. We have

Lemma 14. *We have the following* Spin_7 *decompositions*:

- $\Lambda^2 = \Lambda^2_{(21)} \oplus \Lambda^2_{(7)}$, *corresponding to* $so_8 = \mathrm{spin}_7 \oplus \mathrm{spin}_7^\perp$, *where* $\Lambda^2_{(21)} = \mathrm{spin}_7$ *is the kernel of* $\cdot\Phi : \Lambda^2 \to \Lambda^4,\ \alpha \mapsto \alpha \cdot \Phi$.
- $\Lambda^3 = \Lambda^3_{(8)} \oplus \Lambda^3_{(48)}$, *where* $\Lambda^3_{(48)}$ *is the kernel of* $\Phi\wedge : \Lambda^2 \to \Lambda^6,\ \alpha \mapsto \Phi \wedge \alpha$.
- $\Lambda^4 = \Lambda^4_{(1)} \oplus \Lambda^4_{(7)} \oplus \Lambda^4_{(27)} \oplus \Lambda^4_{(35)}$, *where* $\Lambda^4_{(7)}$ *is the image of* $\cdot\Phi : \Lambda^2_{(7)} \to \Lambda^4,\ \alpha \mapsto \alpha \cdot \Phi$.

For the proof, see e.g. [2]. It follows from this lemma that $W := \Lambda^1 \otimes (\mathrm{spin}_7^\perp \cdot \Phi) = \Lambda^1 \otimes \Lambda^4_{(7)}$, so we decompose

$$W \cong \Lambda^1 \otimes \Lambda^2_{(7)} = W_8 \oplus W_{48},$$

corresponding to the kernel and co-kernel (image of the adjoint) of the interior product map $\mathrm{int} : \Lambda^1 \otimes \Lambda^2_{(7)} \to \Lambda^1$.

6.3. The left-hand side of formula (1)

The 4-form Φ is self-dual. It follows, like in the case of SU_n, that $\|d\Phi\| = \|d^*\Phi\|$, so that

$$\|d\Phi\|^2 = \|d^*\Phi\|^2 = \|\mathrm{int}(\nabla\Phi)\|^2 = a_8\|(\nabla\Phi)_8\|^2 + a_{48}\|(\nabla\Phi)_{48}\|^2,$$

for some homothety factors a_8, a_{48}.

Summand	$w_i \in W_i$	$\|w_i\|^2$	$\|\mathrm{int}(w_i)\|^2$	a_i
W_8	$\sum_{i=1}^7 \theta_i \otimes (\theta_{0i} \cdot \Phi)$	$\frac{7}{3}$	$16 \cdot 7$	48
W_{48}	$\theta_0 \otimes [(\theta_{12} + \theta_{36}) \cdot \Phi] + \theta_1 \otimes [\theta_{20} \cdot \Phi] +$ $\theta_2 \otimes [\theta_{01} \cdot \Phi] + \theta_3 \otimes [\theta_{60} \cdot \Phi] + \theta_6 \otimes [\theta_{03} \cdot \Phi]$	$\frac{4}{3}$	8	6

Table 5: Calculation of the homothety factors a_i for Spin_7

Details of the calculation: To pick a $w_8 \in W_8$, we define an isomorphism $\Lambda^1 \to W_8$ by considering the composition

$$\Lambda^1 \to \Lambda^1 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^4,$$

where the first map is int^* (the adjoint of interior product) and the second is given by the map $\Lambda^2 \to \Lambda^4, \alpha \mapsto \alpha \cdot \Phi$. Starting with θ_0, we have that $\mathrm{int}^*(\theta_0)$ is, up to a constant, $\sum_{i=1}^{7} \theta_i \otimes \theta_{0i}$, and so we get (after some moderate calculation)

$$w_8 = \sum_{i=1}^{7} \theta_i \otimes (\theta_{0i} \cdot \Phi) = \sum_{i=1}^{7} \theta_i \otimes [\theta_i \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_i \otimes (*\phi))],$$

$$\mathrm{int}(w_8) = \sum_{i=1}^{7} \mathrm{int}[\theta_i \otimes (\theta_i \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_i \otimes (*\phi)))]$$

$$= \sum_{i=1}^{7} \mathrm{int}[\theta_i \otimes (\theta_i \wedge \phi)] = 4\phi.$$

To pick a $w_{48} \in W_{48}$ we define an isomorphism $\Lambda^3_{(48)} \to W_{48}$ via the composition

$$\Lambda^3_{(48)} \to \Lambda^3 \to \Lambda^1 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^4,$$

where the first map is inclusion, the second is alt^* (the adjoint of alternation, or exterior product), and the third is given as before by the Λ^2-action on Φ.

To pick an element in $\Lambda^3_{(48)} = \{\alpha \mid \Phi \wedge \alpha = 0\}$, we try an α of the form $\alpha = \theta_0 \wedge \alpha_0$, $\alpha_0 \in \Lambda^2(\mathbb{R}^7)$. Then $\Phi \wedge \alpha = 0$ if and only if $*\phi \wedge \alpha_0 = 0$. The last equation, by a Schur lemma type argument, is equivalent to $\alpha_0 \in \Lambda^2_{(14)}$, i.e. α_0 is in the stabilizer of ϕ (or $*\phi$), from which one obtains easily by inspection a solution such as $\alpha_0 = \theta_{12} + \theta_{36}$, so that $\alpha = \theta_{012} + \theta_{036} \in \Lambda^3_{(48)}$. Now

$$\mathrm{alt}^*(\alpha) = \theta_0 \otimes (\theta_{12} + \theta_{36}) + \theta_1 \otimes \theta_{20} + \theta_2 \otimes \theta_{01} + \theta_3 \otimes \theta_{60} + \theta_6 \otimes \theta_{03},$$

and

$$\theta_{0i} \cdot \Phi = \theta_i \wedge \phi - \theta_0 \wedge \mathrm{int}[\theta_i \otimes (*\phi)],$$

so that

$$\begin{aligned} w_{48} = {} & \theta_2 \otimes [\theta_1 \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_1 \otimes (*\phi))] - \theta_1 \otimes [\theta_2 \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_2 \otimes (*\phi))] \\ & + \theta_6 \otimes [\theta_3 \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_3 \otimes (*\phi))] - \theta_3 \otimes [\theta_6 \wedge \phi - \theta_0 \wedge \mathrm{int}(\theta_6 \otimes (*\phi))], \end{aligned}$$

(note that the $\theta_0 \otimes \alpha_0$ term in $\mathrm{alt}^*(\alpha)$ maps to 0, since $\alpha_0 \cdot \phi = 0$).

Thus $\mathrm{int}(w_{48}) = \cdots = -2\alpha$, where "$\cdots$" denotes a moderately tedious, yet straightforward, calculation.

We thus have,

$$\|d\Phi\|^2 = \|d^*\Phi\|^2 = 48\|(\nabla\Phi)_8\|^2 + 6\|(\nabla\Phi)_{48}\|^2.$$

Remark. It follows from this formula that Φ is parallel if it is closed.

6.4. The right-hand side of formula (1) (*the curvature term*)

The proof of the following lemma is very similar to the G_2-case and is left to the reader:

Lemma 15. *Let M be an 8-dimensional manifold with a* Spin_7*-structure, a 4-form Φ, a Riemann curvature tensor R and scalar curvature s. Then $\langle \widetilde{R}\Phi, \Phi\rangle = 2s$.*

6.5. The Spin_7 *formula*

Denote by E_8, E_{48} the L_2 norms of $(\nabla\Phi)_8$, $(\nabla\Phi)_{48}$ (resp.). Then the above information into formula (1) gives

$$6E_8 - E_{48} = \frac{1}{6}\int s.$$

6.6. Examples

$S^7 \times S^1$. In [4] it is shown that $S^7 \times S^1$ admits an Spin_7-structure of type W_8.

The reader may find there a few more examples.

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